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VARIATIONAL PRINCIPLES IN THE LINEAR THEORY OF VISCOELASTICITY

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1. Introduction

The object of this paper is to supply generalizations to linear quasi-static viscoelasticity theory of certain variational principles which characterize the solution of the mixed boundary-value problem of classical elastostatics. This problem consists in finding a "state" - i.e. a displacement, strain, and stress field - which satisfies the governing field equations in a given region of space and meets the standard mixed boundary conditions. The relevant field equations consist of the displacement-strain relations, the stress-strain relations, and the stress equations of equilibrium; whereas the boundary conditions involve the prescription of displacements over a portion of the boundary and of surface tractions over the remainder.

Two of the most important variational principles applicable to the foregoing problem are the principle of stationary potential energy[1] and the principle of stationary complementary energy.[1] The former asserts that the variation of the "potential energy" over the set of all kinematically admissible states[2] is

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1 See, for example, Sokolnikoff [1](Articles 107, 108). If the elastic constants are such that the strain energy density is a positive definite function of the strains then these variational principles imply corresponding minimum principles.

2 By a kinematically admissible state we mean a state that satisfies the displacement-strain relations, the stress-strain relations, and the displacement boundary conditions.

zero at a certain state _if_ and _only if_ that state is a solution of the mixed problem under consideration.

On the other hand the principle of stationary complementary energy asserts that the variation of the "complementary energy" over the set of all statically admissible stress fields[3] is zero at a certain stress field _if_ that stress field belongs to the solution of the mixed problem. Southwell [2] and Langhaar [3] proved a converse of this theorem on the assumption that the tractions are prescribed over the entire boundary and the region is simply connected: the variation of the "complementary energy" over the set of all statically admissible stress fields is zero at a stress field _only if_ that stress field belongs to the solution of the problem at hand. For the case in which displacements are prescribed over a portion of the boundary a similar converse follows from a slight modification[4] of a theorem due to Dorn and Schild [4].

Various extensions of the preceding variational principles of elastostatics have been established in which the class of admissible states is subjected to weaker restrictions. One extension of this kind was given by Hellinger [5] and was later independently discovered in a somewhat stronger form by Reissner [6],[7].

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[3] By a statically admissible stress field we mean a stress field that meets the stress equations of equilibrium as well as the traction boundary conditions.

[4] See Section 4 for a statement and proof of the modified theorem.

This principle asserts that the variation of a certain functional over the set of all states which meet the strain-displacement relations is zero at a particular state if and only if that state is a solution of the mixed problem. Apparently guided by Reissner's improved version of Hellinger's theorem, Hu Hai-chang [8] and Washizu [9] separately arrived at a still broader variational principle which does not require the admissible states to meet any of the field equations or boundary conditions.

This paper aims at variational principles for linear viscoelasticity which generalize the foregoing results of classical elastostatics. Although variational principles for viscoelasticity theory were considered previously by Biot [10], Freudenthal and Geiringer [11], and Onat [12], these investigations do not arrive at generalizations of the type sought here.

The present paper is a continuation of a recent study [13] which contains a systematic treatment of linear viscoelasticity theory based on the notion of a Stieltjes convolution.

Section 2 contains certain preliminary definitions and notational agreements. In Section 3 variational principles appropriate to the linear quasi-static theory of viscoelastic solids are given for the case in which the stress-strain relations are in relaxation integral form. Section 4 is devoted to the derivation of analogous results for stress-strain relations in creep integral form. In the variational principles established here the viscoelastic solid is allowed to be inhomogeneous and anisotropic and the relevant stress, strain, and displacement histories are permitted to possess finite jump discontinuities in time.

2. Notation. Preliminary definitions.

Throughout what follows R will denote an open region of three-dimensional Euclidean space with the closure $\bar{R}$ and the boundary B. Further $\underline{n}$ will denote the unit outward normal to B and B_α (α=1,2) will denote complementary subsets[5] of B ($B = B_1 \cup B_2$, $B_1 \cap B_2 = 0$). Finally, the symbol "$\times$" will be used to indicate the cartesian product of two sets.

Let $u_i, \varepsilon_{ij}, \sigma_{ij}, F_i, G_{ijk\ell}$, and $J_{ijk\ell}$, in this order, designate the cartesian components of the displacement vector $\underline{u}$, the strain tensor $\underline{\varepsilon}$, the stress tensor $\underline{\sigma}$, the body force (density) vector $\underline{F}$, the relaxation tensor $\underline{G}$, and the creep tensor $\underline{J}$. All of the preceding field histories, including $\underline{G}$ and $\underline{J}$, are to be regarded as functions of position and time defined on $R\times(-\infty,\infty)$. With this notation the complete system of field equations in the linear quasi-static theory of (inhomogeneous and anisotropic) viscoelastic solids take the form[6]

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} \quad \text{on} \quad R\times(-\infty,\infty), \tag{2.1}$$

$$\sigma_{ij,j} + F_i = 0 \ , \ \sigma_{ij} = \sigma_{ji} \quad \text{on} \quad R\times(-\infty,\infty), \tag{2.2}$$

and <u>either</u>

$$\sigma_{ij} = G_{ijk\ell} * d\varepsilon_{k\ell} \quad \text{on} \quad R\times(-\infty,\infty), \tag{2.3}$$

<u>or</u>

$$\varepsilon_{ij} = J_{ijk\ell} * d\sigma_{k\ell} \quad \text{on} \quad R\times(-\infty,\infty). \tag{2.4}$$

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[5] Henceforth the subscript α will be understood to have the range of the integers (1,2).

[6] We use the usual indicial notation. Thus Latin subscripts have the range of the integers (1,2,3) and summation over repeated subscripts is implied; subscripts preceded by a comma indicate differentiation with respect to the corresponding cartesian coordinate.

Equations (2.1) are the linearized strain-displacement relations, (2.2) are the stress equations of equilibrium, (2.3) represent the stress-strain relations in relaxation integral form, while (2.4) represent the stress-strain relations in creep integral form. In writing (2.3), (2.4) we have made use of the notation for Stieltjes convolutions introduced previously in [13]. Thus, if f and g are functions of position and time, $f*dg$ stands for the function defined by the Stieltjes integral

$$[f*dg](\underline{x},t) = \int_{\tau=-\infty}^{t} f(\underline{x},t-\tau)dg(\underline{x},\tau), \tag{2.5}$$

provided this integral is meaningful. To the system of field equations just cited we adjoin the initial conditions

$$\underline{u} = \underline{\varepsilon} = \underline{\sigma} = 0 \quad \text{on} \quad R\times(-\infty, 0), \tag{2.6}$$

the displacement boundary conditions

$$\underline{u} = \hat{\underline{u}} \quad \text{on} \quad B_1\times(-\infty, \infty), \tag{2.7}$$

and the traction boundary conditions

$$\underline{S} = \hat{\underline{S}} \quad \text{on} \quad B_2\times(-\infty, \infty). \tag{2.8}$$

In (2.8) $\underline{S}$ is the surface traction vector with components $S_i=\sigma_{ij}n_j$, while $\hat{\underline{u}}$ and $\hat{\underline{S}}$ are prescribed functions.

The mixed boundary-value problem thus consists in finding field histories $\underline{u},\underline{\varepsilon},\underline{\sigma}$ which, for given R,B_α, known $\underline{G}$ [or $\underline{J}$], and prescribed $\underline{F},\hat{\underline{u}},\hat{\underline{S}}$, satisfy (2.1), (2.2), (2.3) [or (2.4)], (2.6), (2.7), (2.8). We will let $\mathcal{G}=\mathcal{G}(R,B_\alpha,\hat{\underline{u}},\hat{\underline{S}},\underline{F},\underline{G})$ denote the foregoing problem for the case in which the stress-strain relations are in relaxation integral form - i.e. (2.3) holds. On the other hand, if the stress-strain law is given in the creep integral form (2.4), we will denote this problem by $\mathcal{J} = \mathcal{J}(R, B_\alpha, \underline{\hat{u}}, \underline{\hat{S}}, \underline{F}, \underline{J})$.

In order to avoid repeated regularity assumptions concerning the data we define a

Regular problem. We say that $\mathcal{G} = \mathcal{G}(R, B_\alpha, \underline{\hat{u}}, \underline{\hat{S}}, \underline{F}, \underline{G})$ is a regular problem of relaxation type if:

(a) R is a bounded region, its boundary B consists of a finite number of non-intersecting closed regular surfaces,[7] and the subsets B_α are regular surfaces;

(b) (i) $\underline{\hat{u}}$ is a vector-valued function defined on $B_1 \times (-\infty, \infty)$ which vanishes on $B_1 \times (-\infty, 0)$ and is uniformly continuous on $B_1 \times [0, \tau]$ $(0 < \tau < \infty)$;

(ii) $\underline{\hat{S}}$ is a vector-valued function defined on $B_2 \times (-\infty, \infty)$ which vanishes on $B_2 \times (-\infty, 0)$ and is uniformly continuous on $B_2' \times [0, \tau]$ $(0 < \tau < \infty)$ for every regular surface element $B_2' \subset B_2$;

(iii) $\underline{F}$ is a vector-valued function defined on $R \times (-\infty, \infty)$ which vanishes on $R \times (-\infty, 0)$ and is uniformly continuous on $R \times [0, \tau]$ $(0 < \tau < \infty)$.

(c) G is a fourth-order tensor-valued function (of position and time) defined on $\bar{R} \times (-\infty, \infty)$ which vanishes on $\bar{R} \times (-\infty, 0)$, is continuously differentiable on $\bar{R} \times [0, \infty)$, and has the symmetry properties

$$G_{ijk\ell} = G_{jik\ell} = G_{k\ell ij} \quad \text{on} \quad \bar{R} \times (-\infty, \infty). \tag{2.9}$$

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[7] See Kellogg [14] for the definition of a closed regular surface and the definition of a regular surface element.

<u>We say that $\mathcal{J} = \mathcal{J}(R, B_\alpha, \hat{\underline{u}}, \hat{\underline{S}}, \underline{F}, \underline{J})$ is a regular problem of creep type if</u> (a),(b),(c) <u>hold with $\underline{G}$ replaced by $\underline{J}$</u>.

Condition (i) in (b) is equivalent to the requirement that $\hat{\underline{u}}$ coincide on $B_1 \times [0,\tau]$ with a function continuous on the closure of $B_1 \times [0,\tau]$. The analogous comment applies to (iii) in (b). If $\hat{\underline{S}}$ is the prescribed boundary traction of a problem and $\underline{\sigma}$ is the stress tensor of the corresponding solution then $\hat{S}_i = \sigma_{ij} n_j$ on $B_2 \times [0, \infty)$. This motivates condition (ii) in (b) since $\underline{n}$ has finite jump discontinuities on B_2, but not on any regular surface element $B_2' \subset B_2$.

The first of the symmetry relations appearing in (2.9) is a direct consequence of the symmetry of the stress tensor. The second of (2.9), for the special case of an isotropic solid, follows automatically from the condition that the values of $\underline{G}$ be isotropic. For the general <u>anisotropic</u> solid this second symmetry relation constitutes an independent assumption.[8]

Our main objective is the characterization of the solution to the foregoing boundary-value problem by means of variational principles. It thus becomes essential to spell out precisely what we mean by a regular solution to the problem. To this end we first give the following definition of an

<u>Admissible state</u>. <u>We say that the ordered array $\mathcal{S} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}]$ is an admissible state on $\bar{R} \times (-\infty, \infty)$ if</u>:

[8] Theoretical support for this assumption has occasionally been based on thermodynamic arguments involving an appeal to Onsager's principle. See Rogers and Pipkin [15] for a discussion of this issue.

(a) u is a vector-valued function defined on $\bar{R}\times(-\infty,\infty)$, while ε and σ are symmetric second-order tensor-valued functions defined on $\bar{R}\times(-\infty,\infty)$;

(b) u,ε,σ vanish on $\bar{R}\times(-\infty,0)$ and are continuously differentiable on $\bar{R}\times[0,\infty)$.

Note that an admissible state is allowed to have finite jump discontinuities at the time origin and need not meet (2.1), (2.2), (2.3), or (2.4). Addition of states and multiplication of a state by a scalar are defined by

$$\mathcal{S}+\tilde{\mathcal{S}} = [u+\tilde{u},\varepsilon+\tilde{\varepsilon},\sigma+\tilde{\sigma}]\ ,\ \alpha\mathcal{S} = [\alpha u,\alpha\varepsilon,\alpha\sigma]. \qquad (2.10)$$

In view of (2.10) the set of all admissible states on $\bar{R}\times(-\infty,\infty)$ is a linear space.[9]

We are now ready to introduce the notion of a Regular solution. Let $\mathcal{G}=\mathcal{G}(R,B_\alpha,\hat{u},\hat{S},F,G)$ be a regular problem of relaxation type. Then we say that $\mathcal{S}=[u,\varepsilon,\sigma]$ is a regular solution of $\mathcal{G}$ if:

(a) $\mathcal{S}$ is an admissible state on $\bar{R}\times(-\infty,\infty)$;

(b) u,ε,σ meet the field equations (2.1), (2.2), (2.3) and satisfy the boundary conditions (2.7), (2.8).

Let $\mathcal{G}=\mathcal{G}(R,B_\alpha,\hat{u},\hat{S},F,J)$ be a regular problem of creep type. Then we say that $\mathcal{S}=[u,\varepsilon,\sigma]$ is a regular solution of $\mathcal{G}$ if (a), (b) hold with (2.3) replaced by (2.4).

[9] See, for example, Taylor [16] for the definition of a linear space.

Clearly a regular solution is allowed to possess finite jump discontinuities at time zero.

Next we define the

Variation of a functional. *Let* $\Omega\{\cdot\}$ *be a functional defined on a subset* K *of a linear space* L. *Let*

$$\tilde{\mathcal{S}} \in L \;, \mathcal{S}+\alpha\tilde{\mathcal{S}} \in K \quad \text{for every real number } \alpha \;, \tag{2.11}$$

and formally define the notation

$$\delta_{\tilde{\mathcal{S}}}\Omega\{\mathcal{S}\} = \frac{d}{d\alpha}\,\Omega\{\mathcal{S}+\alpha\tilde{\mathcal{S}}\}\Big|_{\alpha=0} \;. \tag{2.12}$$

We say that the variation of $\Omega\{\cdot\}$ *is zero at* $\mathcal{S}$ *and write*

$$\delta\Omega\{\mathcal{S}\} = 0 \quad \text{over} \quad K \tag{2.13}$$

whenever $\delta_{\tilde{\mathcal{S}}}\Omega\{\mathcal{S}\}$ *exists and equals zero for every choice of* $\tilde{\mathcal{S}}$ *consistent with* (2.11).

Unless otherwise specified, the underlying linear space L for the variational principles proved in this paper will be the set of all admissible states on $\bar{R}\times(-\infty,\infty)$.

Finally, we will consistently write $\underline{S}$ and $\underline{\tilde{S}}$ for the traction vectors with components

$$S_i = \sigma_{ij}n_j \;, \; \tilde{S}_i = \tilde{\sigma}_{ij}n_j \tag{2.14}$$

respectively.

3. Variational principles for problems of relaxation type.

We begin with a generalization of the theorem due to the Hu Hai-chang [8] and Washizu [9] mentioned previously.

First variational principle. Let $\mathcal{G}=\mathcal{G}(R,B_\alpha,\underline{\hat{u}},\underline{\hat{S}},\underline{F},\underline{G})$ be a regular problem of relaxation type. Let K be the set of all admissible states on $\bar{R}\times(-\infty,\infty)$. Let $\mathcal{S}=[\underline{u},\underline{\varepsilon},\underline{\sigma}]\in K$ and for each fixed $t\in(-\infty,\infty)$ define the functional $\Lambda_t\{\cdot\}$ on K through

$$\Lambda_t\{\mathcal{S}\} = \frac{1}{2}\int_R [G_{ijk\ell}*d\varepsilon_{ij}*d\varepsilon_{k\ell}](\underline{x},t)dV_{\underline{x}} - \int_R [\sigma_{ij}*d\varepsilon_{ij}](\underline{x},t)dV_{\underline{x}}$$
$$- \int_R [(\sigma_{ij,j}+F_i)*du_i](\underline{x},t)dV_{\underline{x}} + \int_{B_1} [S_i*d\hat{u}_i](\underline{x},t)dA_{\underline{x}}$$
$$+ \int_{B_2} [(S_i-\hat{S}_i)*du_i](\underline{x},t)dA_{\underline{x}} . \qquad (3.1)^{10}$$

Then

$$\delta\Lambda_t\{\mathcal{S}\} = 0 \ \text{over} \ K \ (-\infty < t < \infty) \qquad (3.2)$$

if and only if $\mathcal{S}$ is a regular solution of $\mathcal{G}$.

Proof: Let $\tilde{\mathcal{S}} = [\underline{\tilde{u}},\underline{\tilde{\varepsilon}},\underline{\tilde{\sigma}}]\in K$ from which it follows that $\mathcal{S}+\alpha\tilde{\mathcal{S}}\in K$. Then by (2.12), (2.9), Theorems 1.2 and 1.6 of [13], and the symmetry of $\underline{\tilde{\sigma}}$

[10] We write $dV_{\underline{x}}$ and $dA_{\underline{x}}$ for the volume element and element of area, respectively, to indicate that $\underline{x}$ is the variable of integration.

$$\delta_{\tilde{\mathcal{S}}}\Lambda_t\{\mathcal{S}\} = \int_R [(G_{ijk\ell}*d\varepsilon_{k\ell}-\sigma_{ij})*d\tilde{\varepsilon}_{ij}](\underline{x},t)dV_{\underline{x}} +$$
$$- \int_R [(\sigma_{ij,j}+F_i)*d\tilde{u}_i](\underline{x},t)dV_{\underline{x}} +$$
$$- \int_R [(\varepsilon_{ij}-\tfrac{1}{2}(u_{i,j}+u_{j,i}))*d\tilde{\sigma}_{ij}](\underline{x},t)dV_{\underline{x}} +$$
$$+ \int_{B_1} [(\hat{u}_i-u_i)*d\tilde{S}_i](\underline{x},t)dA_{\underline{x}} +$$
$$+ \int_{B_2} [(S_i-\hat{S}_i)*d\tilde{u}_i](\underline{x},t)dA_{\underline{x}} \quad (-\infty < t < \infty). \tag{3.3}$$

First suppose $\mathcal{S}$ is a solution of $\mathcal{G}$. Then by virtue of (2.1), (2.2),(2.3),(2.7),(2.8), equation (3.3) becomes

$$\delta_{\tilde{\mathcal{S}}}\Lambda_t\{\mathcal{S}\} = 0 \quad \text{over} \quad K \quad (-\infty < t < \infty). \tag{3.4}$$

Equation (3.2) now follows from (3.4) since $\tilde{\mathcal{S}} \in K$ was chosen arbitrarily.

Now turn to the "only if" portion of the proof. We must show that $\mathcal{S}$ is a regular solution of $\mathcal{G}$ whenever $\mathcal{S} \in K$ and (3.4) holds for every $\tilde{\mathcal{S}} \in K$. In particular choose

$$\underline{\tilde{u}}(\underline{x},t) = \underline{u}'(\underline{x})h(t),\ \underline{\tilde{\varepsilon}}(\underline{x},t) = \underline{\varepsilon}'(\underline{x})h(t),\ \underline{\tilde{\sigma}}(\underline{x},t) = \underline{\sigma}'(\underline{x})h(t) \tag{3.5}$$

for every $(\underline{x},t) \in \bar{R}\times(-\infty,\infty)$, where h is the Heaviside unit step function, i.e. $h(t) = 0\ (-\infty < t < 0)$, $h(t) = 1\ (0 \le t < \infty)$. Therefore (3.4), by virtue of (3.3) and Theorem 1.2 in [13], becomes

$$\int_R [G_{ijk\ell} * d\varepsilon_{k\ell} - \sigma_{ij}](\underline{x},t)\varepsilon'_{ij}(\underline{x})dV_{\underline{x}} - \int_R [\sigma_{ij,j} + F_i](\underline{x},t)u'_i(\underline{x})dV_{\underline{x}} +$$

$$- \int_R [\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})](\underline{x},t)\sigma'_{ij}(\underline{x})dV_{\underline{x}} +$$

$$+ \int_{B_1} [\hat{u}_i - u_i](\underline{x},t)\sigma'_{ij}(\underline{x})n_j(\underline{x})dA_{\underline{x}} +$$

$$+ \int_{B_2} [S_i - \hat{S}_i](\underline{x},t)u'_i(\underline{x})dA_{\underline{x}} = 0 \quad (-\infty < t < \infty) \tag{3.6}$$

and (3.6) must hold for every $\underline{u}', \underline{\varepsilon}', \underline{\sigma}'$ continuously differentiable on $\overline{R}$ with $\underline{\varepsilon}'$ and $\underline{\sigma}'$ symmetric. But this fact, the fundamental lemma of the calculus of variations, and the symmetries of $\underline{\sigma}, \underline{\varepsilon}, \underline{G}$ imply that $\mathcal{S}$ meets (2.1),(2.2),(2.3),(2.8) and that

$$\int_{B_1} [\hat{u}_i - u_i](\underline{x},t)\sigma'_{ij}(\underline{x})n_j(\underline{x})dA_{\underline{x}} = 0 \quad (-\infty < t < \infty). \tag{3.7}$$

Now assume there exists a regular point $\underline{x}^o \in B_1$ such that $\underline{u}(\underline{x}^o,t) \neq \hat{\underline{u}}(\underline{x}^o,t)$ which implies $u_k(\underline{x}^o,t) \neq \hat{u}_k(\underline{x}^o,t)$ for some (fixed) k. Choose the coordinate frame such that $n_k(\underline{x}^o) \neq 0$, let f be continuously differentiable on $\overline{R}$, and let $\sigma'_{ij}(\underline{x}) = \delta_{ik}\delta_{jk}f(\underline{x})$ (no sum). Then, since (3.7) must hold for every such $\underline{\sigma}'$, $[u_k(\underline{x},t) - \hat{u}_k(\underline{x},t)]n_k(\underline{x}) = 0$ (no sum) for every $(\underline{x},t) \in B_1 \times (-\infty, \infty)$ with $\underline{x}$ regular, which implies $u_k(\underline{x}^o,t) = \hat{u}_k(\underline{x}^o,t)$. Consequently, we have a contradiction and hence $\underline{u}(\underline{x},t) = \hat{\underline{u}}(\underline{x},t)$ for every $(\underline{x},t) \in B_1 \times (-\infty, \infty)$ with $\underline{x}$ regular. Thus and by the continuity of $\underline{u}$ and $\hat{\underline{u}}$, (2.7) holds as well and $\mathcal{S}$ is a solution of $\mathcal{G}$. This completes the proof.

By virtue of the divergence theorem, Λ_t admits the alternative representation

$$\Lambda_t\{\mathcal{S}\} = \frac{1}{2}\int_R [G_{ijk\ell}*d\varepsilon_{ij}*d\varepsilon_{k\ell}](\underline{x},t)dV_{\underline{x}} - \int_R [\sigma_{ij}*d(\varepsilon_{ij}-u_{i,j})](\underline{x},t)dV_{\underline{x}} +$$

$$- \int_R [F_i*du_i](\underline{x},t)dV_{\underline{x}} - \int_{B_1} [S_i*d(u_i-\hat{u}_i)]dA_{\underline{x}} +$$

$$- \int_{B_2} [\hat{S}_i*du_i](\underline{x},t)dA_{\underline{x}} . \tag{3.8}$$

If, in addition to merely being admissible, $\mathcal{S} = [\underline{u},\underline{\varepsilon},\underline{\sigma}]$ meets (2.1),(2.3), and (2.7), then $\Lambda_t\{\mathcal{S}\}$ given by (3.8) reduces to $\Phi_t\{\mathcal{S}\}$, where

$$\Phi_t\{\mathcal{S}\} = \frac{1}{2}\int_R [G_{ijk\ell}*d\varepsilon_{ij}*d\varepsilon_{k\ell}](\underline{x},t)dV_{\underline{x}} - \int_R [F_i*du_i](\underline{x},t)dV_{\underline{x}} +$$

$$- \int_{B_2} [\hat{S}_i*du_i](\underline{x},t)dA_{\underline{x}} . \tag{3.9}$$

Thus we are led to the following generalization of the principle of stationary potential energy.

<u>Second variational principle</u>. <u>Let</u> $\mathcal{G} = \mathcal{G}(R,B_\alpha,\underline{\hat{u}},\underline{\hat{S}},\underline{F},\underline{G})$ <u>be a regular problem of relaxation type</u>. <u>Let</u> K <u>be the set of all admissible states on</u> $\bar{R}\times(-\infty,\infty)$ <u>which meet the strain-displacement relations</u> (2.1), <u>the stress-strain relations</u> (2.3), <u>as well as the displacement boundary conditions</u> (2.7). <u>Let</u> $\mathcal{S} = [\underline{u},\underline{\varepsilon},\underline{\sigma}] \in K$ <u>and for each fixed</u> $t \in (-\infty,\infty)$ <u>define the functional</u> $\Phi_t\{\cdot\}$ <u>on</u> K <u>through</u> (3.9). <u>Then</u>

$$\delta\Phi_t\{\mathcal{S}\} = 0 \quad \underline{\text{over}} \quad K \quad (-\infty < t < \infty) \tag{3.10}$$

<u>if and only if</u> $\mathcal{S}$ <u>is a regular solution of</u> $\mathcal{G}$.

Proof: The "if" portion of the proof follows at once from the first variational principle and the discussion preceding (3.9).

To establish the remainder of the theorem assume

$$\delta_{\tilde{\mathcal{S}}}\Phi_t\{\mathcal{S}\} = 0 \quad \text{over} \quad K \quad (-\infty < t < \infty) \tag{3.11}$$

for every $\tilde{\mathcal{S}}$ which meets (2.11).[11] This latter condition is equivalent to the requirement that $\tilde{\mathcal{S}}$ be admissible and meet (2.1),(2.3), with

$$\underline{\tilde{u}} = 0 \quad \text{on} \quad B_1\times(-\infty, \infty). \tag{3.12}$$

Clearly, (3.3) holds if we replace $\Lambda_t\{\mathcal{S}\}$ by $\Phi_t\{\mathcal{S}\}$ and omit the first, third, and fourth terms, since $\mathcal{S}$ meets (2.1),(2.3), (2.7). Now choose $\underline{\tilde{u}}(\underline{x},t) = \underline{u}'(\underline{x})h(t)$ for every $(\underline{x},t)$ in $\bar{R}\times(-\infty,\infty)$, where h is the Heaviside unit step function and $\underline{u}'$ is twice continuously differentiable on $\bar{R}$, with

$$\underline{u}' = 0 \quad \text{on} \quad B_1 . \tag{3.13}$$

Next define continuously differentiable functions $\underline{\tilde{\varepsilon}},\underline{\tilde{\sigma}}$ on $\bar{R}$ through (2.1),(2.3). Thus and by (3.11),

$$-\int_R [\sigma_{ij,j}+F_i](\underline{x},t)u_i'(\underline{x})dV_{\underline{x}} + \int_{B_2} [S_i-\hat{S}_i](\underline{x},t)u_i'(\underline{x})dA_{\underline{x}} = 0 \qquad (-\infty < t < \infty) \tag{3.14}$$

for every function $\underline{u}'$ with the foregoing properties. But this fact, by virtue of the fundamental lemma of the calculus of variations, implies that $\mathcal{S}$ meets (2.2),(2.8) and the proof is complete.

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[11] Recall our agreement that L is the set of all admissible states on $\bar{R}\times(-\infty,\infty)$.

4. Variational principles for problems of creep type.

The following theorem is a generalization of the Hellinger-Reissner principle in linear elastostatics.

Third variational principle. *Let* $\mathcal{P} = \mathcal{P}(R, B_\alpha, \underline{\hat{u}}, \underline{\hat{S}}, \underline{F}, \underline{J})$ *be a regular problem of creep type. Let* K *be the set of all admissible states on* $\bar{R}\times(-\infty, \infty)$ *which meet the strain-displacement relations* (2.1). *Let* $\mathcal{S} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}] \in K$ *and for each fixed* $t \in (-\infty, \infty)$ *define the functional* $\Theta_t\{\cdot\}$ *on* K *through*

$$\Theta_t\{\mathcal{S}\} = \int_R [\sigma_{ij}*d\varepsilon_{ij}](\underline{x},t)dV_{\underline{x}} - \frac{1}{2}\int_R [J_{ijk\ell}*d\sigma_{ij}*d\sigma_{k\ell}](\underline{x},t)dV_{\underline{x}}$$
$$- \int_R [F_i*du_i](\underline{x},t)dV_{\underline{x}} - \int_{B_1} [S_i*d(u_i-\hat{u}_i)](\underline{x},t)dA_{\underline{x}}$$
$$- \int_{B_2} [\hat{S}_i*du_i](\underline{x},t)dA_{\underline{x}} . \tag{4.1}$$

Then

$$\delta\Theta_t\{\mathcal{S}\} = 0 \quad \text{over} \quad K \quad (-\infty < t < \infty) \tag{4.2}$$

if and only if $\mathcal{S}$ *is a regular solution of* $\mathcal{P}$.

Proof: Let $\tilde{\mathcal{S}} = [\underline{\tilde{u}}, \underline{\tilde{\varepsilon}}, \underline{\tilde{\sigma}}]$ meet (2.11).[12] Then from the definition of K and since $\mathcal{S} \in K$, we have that $\tilde{\mathcal{S}} \in K$. Consequently, because of Theorems 1.2 and 1.6 of [13],

$$\delta_{\tilde{\mathcal{S}}}\Theta_t\{\mathcal{S}\} = \int_R [(\varepsilon_{ij}-J_{ijk\ell}*d\sigma_{k\ell})*d\tilde{\sigma}_{ij}](\underline{x},t)dV_{\underline{x}}$$
$$- \int_R [(\sigma_{ij,j}+F_i)*d\tilde{u}_i](\underline{x},t)dV_{\underline{x}} + \int_{B_1} [\tilde{S}_i*d(\hat{u}_i-u_i)](\underline{x},t)dA_{\underline{x}}$$
$$+ \int_{B_2} [(S_i-\hat{S}_i)*d\tilde{u}_i](\underline{x},t)dA_{\underline{x}} \quad (-\infty < t < \infty). \tag{4.3}$$

[12] See the previous footnote.

The conclusion now follows from (4.3) by an argument which is strictly analogous to that which led from (3.3) to the final conclusion in the proof of the first variational principle.

We turn next to a generalization of the principle of stationary complementary energy in elastostatics. With a view toward an economical statement of this generalized principle we introduce the subsequent notions.

Convexity of R with respect to B_1. We say that R is convex with respect to B_1 if the straight line

$$\underline{x}(\tau) = \hat{\underline{x}} + (\tilde{\underline{x}} - \hat{\underline{x}})\tau \quad (-\infty < \tau < \infty) \tag{4.4}$$

intersects B only at $\hat{\underline{x}}$ and $\tilde{\underline{x}}$ whenever $\hat{\underline{x}}, \tilde{\underline{x}} \in B_1$.

Notice that if $B = B_2$ then R is automatically convex with respect to B_1.

Admissible stress field. We say that σ is an admissible stress field on $\bar{R}\times(-\infty,\infty)$ if σ is a symmetric second-order tensor-valued function defined on $\bar{R}\times(-\infty,\infty)$, which vanishes on $\bar{R}\times(-\infty,0)$ and is continuously differentiable on $\bar{R}\times[0,\infty)$.

Finally, we stipulate that the underlying linear space L for the following theorem is the set of all admissible stress fields on $\bar{R}\times(-\infty,\infty)$.

Fourth variational principle. Let $\mathcal{J} = \mathcal{J}(R, B_\alpha, \hat{\underline{u}}, \hat{\underline{S}}, \underline{F}, \underline{J})$ be a regular problem of creep type. Let K be the set of all admissible stress-fields on $\bar{R}\times(-\infty,\infty)$ which meet the stress equations of equilibrium (2.2) and the traction boundary conditions (2.8). Let $\sigma \in K$ and for each fixed $t \in (-\infty,\infty)$ define the functional $\Psi_t\{\cdot\}$ on K through

$$\Psi_t\{\underline{\sigma}\} = \frac{1}{2}\int_R [J_{ijk\ell}*d\sigma_{ij}*d\sigma_{k\ell}](\underline{x},t)dV_{\underline{x}} - \int_{B_1}[S_i*d\hat{u}_i](\underline{x},t)dA_{\underline{x}}. \quad (4.5)$$

Then

$$\delta\Psi_t\{\underline{\sigma}\} = 0 \quad \text{over} \quad K \quad (-\infty < t < \infty) \quad (4.6)$$

if there exist functions $\underline{u},\underline{\varepsilon}$ such that $[\underline{u},\underline{\varepsilon},\underline{\sigma}]$ is a regular solution of $\mathcal{J}$.

Conversely, suppose

(a) R is convex with respect to B_1;

(b) R is simply-connected;

(c) J and σ are twice continuously differentiable on $\bar{R}\times[0,\infty)$;

(d) $\hat{\underline{u}}(\underline{x},\cdot)$, for each $\underline{x}\in B_1$, is continuously differentiable on $[0,\infty)$;

(e) (4.6) holds.

Then there exist functions $\underline{u},\underline{\varepsilon}$ such that $[\underline{u},\underline{\varepsilon},\underline{\sigma}]$ is a regular solution of $\mathcal{J}$.

Proof: Let $\tilde{\underline{\sigma}}\in L$, $\underline{\sigma}+\alpha\tilde{\underline{\sigma}}\in K$ for every real α. Then

$$\left.\begin{aligned} \tilde{\sigma}_{ij,j} &= 0 \quad \text{on} \quad R\times(-\infty,\infty), \\ \tilde{S}_i \equiv \tilde{\sigma}_{ij}n_j &= 0 \quad \text{on} \quad B_2\times(-\infty,\infty). \end{aligned}\right\} \quad (4.7)$$

Further, since $J_{ijk\ell} = J_{k\ell ij}$, it follows that

$$\delta_{\tilde{\underline{\sigma}}}\Psi_t\{\underline{\sigma}\} = \int_R [J_{ijk\ell}*d\sigma_{k\ell}*d\tilde{\sigma}_{ij}](\underline{x},t)dV_{\underline{x}} - \int_{B_1}[\tilde{S}_i*d\hat{u}_i](\underline{x},t)dA_{\underline{x}}$$
$$(-\infty < t < \infty). \quad (4.8)$$

Now suppose there exist functions $\underline{u}$ and $\underline{\varepsilon}$ such that $[\underline{u},\underline{\varepsilon},\underline{\sigma}]$ is a solution of $\mathcal{J}$. Then, by virtue of (4.7) and the divergence theorem, equation (4.8) implies

$$\delta_{\tilde{\underline{\sigma}}} \Psi_t\{\underline{\sigma}\} = 0 \quad (-\infty < t < \infty). \tag{4.9}$$

Thus, and since $\tilde{\underline{\sigma}}$ was chosen arbitrarily, (4.6) holds.

We turn next to the proof of the converse assertion. To this end we state and prove the following trivial modification of a theorem due to Dorn and Schild [4].

Lemma. Let R be simply-connected and convex with respect to B_1. Let $\hat{u}$ be a vector-valued function which is uniformly continuous on B_1 and let ε be a symmetric second-order tensor-valued function which is twice continuously differentiable on $\bar{R}$. Further suppose

$$\int_R \sigma_{ij}(\underline{x})\varepsilon_{ij}(\underline{x})dA_{\underline{x}} = \int_{B_1} \sigma_{ij}(\underline{x})\hat{u}_i(\underline{x})n_j(\underline{x})dA_{\underline{x}} \tag{4.10}$$

for every symmetric second-order tensor-valued function σ which is continuously differentiable arbitrarily often on $\bar{R}$ and meets

$$\left.\begin{aligned} \sigma_{ij,j} &= 0 \quad \text{on } R, \\ \sigma_{ij}n_j &= 0 \quad \text{on } B_2. \end{aligned}\right\} \tag{4.11}$$

Then there exists a vector-valued function u which is continuously differentiable on $\bar{R}$ and satisfies

$$\left.\begin{aligned} 2\varepsilon_{ij} &= u_{i,j}+u_{j,i} \quad \text{on } R \\ u_i &= \hat{u}_i \quad \text{on } B_1 \end{aligned}\right\} \tag{4.12}$$

Proof: Here we follow the argument of Dorn and Schild. Let $\underline{g}$ be a symmetric second-order tensor-valued function which is continuously differentiable arbitrarily often on $\bar{R}$ and which vanishes identically outside a closed subregion of R. Let γ_{ijk}

denote the usual alternating symbol and define $\underline{\sigma}$ through

$$\sigma_{ij} = \gamma_{ipq}\gamma_{jrs}g_{pr,qs} \quad \text{on} \quad R, \tag{4.13}$$

i.e. use $\underline{g}$ as a Gwyther-Finzi stress function.[13] This choice of $\underline{\sigma}$ meets (4.11), has the requisite degree of smoothness, and vanishes on B. Therefore (4.10) implies

$$\int_R \varepsilon_{ij}\gamma_{ipq}\gamma_{jrs}g_{pr,qs}dV = 0 \ . \tag{4.14}$$

Now integrate (4.14) twice by parts and use the fact that $\underline{g}$ and all of its partial derivatives vanish on B to deduce that

$$\int_R (\gamma_{ipq}\gamma_{jrs}\varepsilon_{ij,qs})g_{pr}dV = 0 \ . \tag{4.15}$$

Since (4.15) must hold for every such function $\underline{g}$

$$\gamma_{piq}\gamma_{rjs}\varepsilon_{ij,qs} = 0 \quad \text{on} \quad R \ . \tag{4.16}$$

Hence $\underline{\varepsilon}$ is a compatible strain field and from the simple-connectivity of R we conclude that there exists a vector-valued function $\underline{u}'$ which is continuously differentiable on $\bar{R}$ and meets

$$2\varepsilon_{ij} = u'_{i,j} + u'_{j,i} \quad \text{on} \quad R \ . \tag{4.17}$$

Moreover, such a $\underline{u}'$ is given by the line integrals[14]

$$u'_i(\underline{x}) = \int_{\underline{x}^o}^{\underline{x}} U_{ij}(\underline{\xi},\underline{x})d\xi_j \quad \text{for every } \underline{x}\in\bar{R} \ , \tag{4.18}$$

where $\underline{x}^o\in R$ and for every $(\underline{\xi},\underline{x})\in \bar{R}\times\bar{R}$

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[13] See, for instance, Truesdell and Toupin [17] (Article 227).

[14] See, for example, Sokolnikoff [1] (Article 10).

$$U_{ij}(\underline{\xi},\underline{x}) = \varepsilon_{ij}(\underline{\xi}) + (x_k-\xi_k)[\varepsilon_{ij,k}(\underline{\xi}) - \varepsilon_{kj,i}(\underline{\xi})]. \quad (4.19)$$

Now let

$$v_i = \hat{u}_i - u_i' \quad \text{on} \quad B_1 \quad (4.20)$$

and use (4.10),(4.11),(4.12), together with the divergence theorem, to establish that

$$\int_{B_1} \sigma_{ij} n_j v_i dA = 0 \; . \quad (4.21)$$

Our next step will be to show that $\underline{v}$ is a rigid displacement field. To this end let $\hat{\underline{x}}$ and $\tilde{\underline{x}}$ be arbitrary interior points of B_1 and choose the coordinate system such that

$$\hat{\underline{x}} = (0,0,0) \; , \; \tilde{\underline{x}} = (0,0,\tilde{x}_3). \quad (4.22)$$

Let D_ε be a disc in the x_1,x_2-plane with radius ε and center at $x_1 = x_2 = 0$ and let

$$\sigma_{ij}(x_1,x_2,x_3) = \delta_{i3}\delta_{j3}f_\varepsilon(x_1,x_2), \quad (4.23)$$

where f_ε is defined on the entire x_1,x_2-plane and has the following properties:

$$\left.\begin{array}{ll} \text{(a)} & f_\varepsilon \text{ is differentiable arbitrarily often;} \\ \text{(b)} & f_\varepsilon \geq 0; \\ \text{(c)} & f_\varepsilon = 0 \quad \text{outside} \quad D_\varepsilon; \\ \text{(d)} & \int_{D_\varepsilon} f_\varepsilon dA = 1 \; . \end{array}\right\} \quad (4.24)$$

Clearly such a $\underline{\sigma}$ meets the first of (4.11). Now let C_ε be the solid circular cylinder whose axis coincides with the x_3-axis and whose cross-section is D_ε. By the assumed convexity of R with respect to B_1

$$(C_\varepsilon \cap B) \subset B_1 \ , \tag{4.25}$$

for sufficiently small ε (say $\varepsilon < \varepsilon_o$). Thus and by (4.23), (4.24), the second of (4.11) holds for $\varepsilon < \varepsilon_o$. Further for $\varepsilon < \varepsilon_1 \leq \varepsilon_o$ there exist disjoint subregions $\hat{\mathcal{B}}_\varepsilon, \tilde{\mathcal{B}}_\varepsilon$ of B_1 such that

$$\underline{\hat{x}} \in \hat{\mathcal{B}}_\varepsilon \ , \ \underline{\tilde{x}} \in \tilde{\mathcal{B}}_\varepsilon \ , \ C_\varepsilon \cap B_1 = \hat{\mathcal{B}}_\varepsilon \cup \tilde{\mathcal{B}}_\varepsilon . \tag{4.26}$$

Consequently, by virtue of (4.23), (4.24c), equation (4.21) reduces to

$$\int_{\hat{\mathcal{B}}_\varepsilon} f_\varepsilon v_3 n_3 dA + \int_{\tilde{\mathcal{B}}_\varepsilon} f_\varepsilon v_3 n_3 dA = 0 \quad (\varepsilon < \varepsilon_1). \tag{4.27}$$

Next let $\underline{m} = (0,0,1)$ be the unit normal vector of D_ε and conclude from (4.23), (4.24d) that

$$\left.\begin{aligned} \int_{\tilde{\mathcal{B}}_\varepsilon} f_\varepsilon n_3 dA &= \int_{\tilde{\mathcal{B}}_\varepsilon} \sigma_{ij} n_j dA = \int_{D_\varepsilon} \sigma_{ij} m_j dA = \int_{D_\varepsilon} f_\varepsilon dA = 1 \ , \\ \int_{\hat{\mathcal{B}}_\varepsilon} f_\varepsilon n_3 dA &= \int_{\hat{\mathcal{B}}_\varepsilon} \sigma_{ij} n_j dA = - \int_{D_\varepsilon} \sigma_{ij} m_j dA = - \int_{D_\varepsilon} f_\varepsilon dA = -1 \ , \end{aligned}\right\} \tag{4.28}$$

provided $\varepsilon < \varepsilon_1$. Now let $\varepsilon \to 0$ in (4.27) and use (4.24b), (4.28) to infer that

$$v_3(\underline{\hat{x}}) - v_3(\underline{\tilde{x}}) = 0. \tag{4.29}$$

But (4.29), because of (4.22) and since $\underline{\hat{x}}$, $\underline{\tilde{x}}$ were chosen arbitrarily, implies

$$[\underline{v}(\underline{\hat{x}}) - \underline{v}(\underline{\tilde{x}})] \cdot [\underline{\hat{x}} - \underline{\tilde{x}}] = 0 \tag{4.30}$$

for every $\underline{\hat{x}}$, $\underline{\tilde{x}} \in B_1$. Hence $\underline{v}$ on B_1 must belong to the moment field of a bound vector system and thus admits the representation[15]

[15] See, for example, Nielsen [18] (Chapter 3).

$$v_i(\underline{x}) = a_i + \omega_{ij}x_j \quad (a_i, \omega_{ij} = -\omega_{ji} \ldots\ldots \text{constant}) \tag{4.31}$$

for $\underline{x} \in B_1$. Now define $\underline{u}$ on $\overline{R}$ through

$$u_i(\underline{x}) = u_i'(\underline{x}) + a_i + \omega_{ij}x_j \quad \text{if } \underline{x} \in \overline{R} \tag{4.32}$$

and conclude from (4.17), (4.20), (4.31) that $\underline{u}$ meets (4.12). This completes the proof of the lemma.

We turn now to the remainder of the proof of the fourth variational principle. To this end suppose hypotheses (a) through (e) hold. Clearly, (4.8) is satisfied by every admissible stress field $\tilde{\underline{\sigma}}$ which meets (4.7). In particular let

$$\tilde{\sigma}_{ij}(\underline{x},t) = \sigma_{ij}'(\underline{x})h(t) \quad \text{for every } (\underline{x},t) \in \overline{R}\times(-\infty,\infty). \tag{4.33}$$

Next define $\underline{\varepsilon}$ on $\overline{R}\times(-\infty,\infty)$ through

$$\varepsilon_{ij} = J_{ijk\ell} * d\sigma_{k\ell} \tag{4.34}$$

and observe that hypothesis (c) and Theorem 1.6 of [13] imply that $\underline{\varepsilon}$ vanishes on $\overline{R}\times(-\infty,0)$ and is twice continuously differentiable on $\overline{R}\times[0,\infty)$. Further, infer from (4.6), (4.8), (4.33), (4.34) that

$$\int_R \sigma_{ij}'(\underline{x})\varepsilon_{ij}(\underline{x},t)dV_{\underline{x}} = \int_{B_1} \sigma_{ij}'(\underline{x})\hat{u}_i(\underline{x},t)n_j(\underline{x})dA_{\underline{x}} \quad (-\infty < t < \infty) \tag{4.35}$$

for every $\underline{\sigma}'$ which is twice continuously differentiable on $\overline{R}$ and meets

$$\left.\begin{aligned} \sigma_{ij,j}' &= 0 \quad \text{on} \quad R, \\ \sigma_{ij}'n_j &= 0 \quad \text{on} \quad B_2. \end{aligned}\right\} \tag{4.36}$$

Equations (4.35), (4.36), together with the preceding lemma imply the existence of a displacement field $\underline{u}$ which satisfies (2.1), (2.7). Moreover, it is clear from the smoothness of $\underline{\varepsilon}$, hypothesis (d), and the proof of the lemma that $\underline{u}$ vanishes on $\bar{R}\times(-\infty,0)$, and is continuously differentiable on $\bar{R}\times[0,\infty)$. Thus we have shown that $[\underline{u},\underline{\varepsilon},\underline{\sigma}]$ is a regular solution of $\mathcal{J}$ and the proof is complete.

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